J. Fluid Mech. (1974), *vol.* 66, *part* 2, *pp.* 261–265
Printed in Great Britain

Expansions for the shape of maximum amplitude Stokes waves

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(Received 28 January 1974)

Grant (1973) showed that the expansion giving the profile of a steady Stokes wave near a 120° corner was more complicated than had previously been assumed. This paper gives further terms in such an expansion, and shows that generating them cannot introduce transcendental quantities beyond those noted by Grant.

1. Introduction

This note is concerned with producing an expansion giving the profile of a symmetric, steady, progressing, two-dimensional wave. The first term in the expansion is provided by Stokes' solution to the equations involved, which shows that if the wave crest has a corner it must be of 120°. Expressed in terms of streamline co-ordinates, such a corner corresponds to a function with a singularity of order $\frac{2}{3}$. Grant (1973) considered possible subsequent terms in the expansion of this function, and showed that they must have transcendental exponents, making the singularity irregular. This, being contrary to the assumption of previous workers, casts doubt on earlier results. Using an experimental computer program SCRATCHPAD (Griesmer & Jenks 1972) to do some of the formal algebraic manipulation involved, further terms in the expansion Grant started have been produced, and it can be seen that the only irrational quantities that can arise are those that appear right at the beginning.

2. Equations

With respect to streamline co-ordinates (ϕ, ψ) with $f = \phi + i\psi$, and space co-ordinates (x, y) with $z = x + iy$, the wave profile desired can be found from the solution for $z(f)$ of

$$-2\,\mathrm{Im}\,z\,|dz/df|^2 = 1 \quad \text{at} \quad \mathrm{Im}\,f = 0. \tag{1}$$

The derivation of this equation, which represents the Bernoulli condition on the wave, can be found in Grant (1973).

Stokes' solution of (1) is

$$z = -i(\tfrac{3}{2}if)^{\frac{2}{3}}$$

and so we look for a solution of the form

$$z = -i(\tfrac{3}{2}if)^{\frac{2}{3}}\,w,$$

where $w = u + iv$.

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Since (1) is to be applied for real f only, f can be treated as a real variable, and so splitting into real and imaginary parts gives

$$x = \tfrac{1}{2}(\tfrac{3}{2}f)^{\frac{2}{3}}(3^{\frac{1}{2}}u+v), \quad y = \tfrac{1}{2}(\tfrac{3}{2}f)^{\frac{2}{3}}(u-3^{\frac{1}{2}}v)$$

(using $i^{\frac{2}{3}} = \cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi = \frac{1}{2}+\frac{1}{2}i3^{\frac{1}{2}}$). It is now helpful to define

$$\dot{u} = u + \tfrac{3}{2}f\,du/df, \quad \dot{v} = v + \tfrac{3}{2}f\,dv/df,$$

then (1) becomes

$$(u-3^{\frac{1}{2}}v)(\dot{u}^2+\dot{v}^2) = 1. \tag{2}$$

All the terms in $(\frac{3}{2}f)^{\frac{1}{3}}$ have vanished, and most of the $\sqrt{3}$'s have cancelled. Note as a check that $u = 1, v = 0$ is an exact solution of (2), corresponding to $w = 1$, and the unperturbed original Stokes solution.

3. Expansion

Now try substituting a power series of the form

$$\sum_{j=0}^{\infty} a_j (if)^{2\mu j} \tag{3}$$

for w. The quantity μ will be determined by solving an indicial equation, and is not necessarily an integer. Making the coefficients a_j all real corresponds to considering the expansion of a symmetric wave; $a_0 = 1$ makes the leading term of the expansion correct. Introducing a new variable

$$\eta = f^{2\mu}$$

so that $d\eta/df = 2\mu\eta/f$, and using $i^{2\mu j} = \cos\pi\mu j + i\sin\pi\mu j$, equation (3) implies that

$$u_j = a_j\cos\pi\mu j, \quad v_j = a_j\sin\pi\mu j, \tag{4}$$

where u_j and v_j represent the coefficients of η^j in u and v. Differentiating with respect to f shows that

$$\dot{u}_j = a_j(1+3\mu j)\cos\pi\mu j, \quad \dot{v}_j = a_j(1+3\mu j)\sin\pi\mu_j. \tag{5}$$

When (4) and (5) are used to express (2) in terms of the a_i's all explicit mention of f disappears. Equation (2) reduces to the form

$$\epsilon_0 + \epsilon_1\eta + \epsilon_2\eta^2 + \ldots = 1, \tag{6}$$

where the ϵ_i are polynomials in the a_j, μ and sines and cosines of multiples of $\pi\mu$. With $a_0 = 1$, $\epsilon_0 = 1$ and so the constant parts of (6) vanish. Picking out the coefficient of η in (6) gives

$$\epsilon_1 \equiv a_1[3^{\frac{1}{2}}(2\mu+1)\cos\pi\mu - \sin\pi\mu] = 0.$$

With $a_1 \neq 0$, this means that

$$\tan\pi\mu = 3^{\frac{1}{2}}(2\mu+1). \tag{7}$$

This indicial equation for μ is equivalent to the transcendental equation Grant found, and the fact that it has no rational solutions is the reason why previous expansions of the Stokes wave profile have been invalid. It should be noted that solving $\epsilon_1 = 0$ puts no constraints on a_1.

The coefficients $a_2, a_3, \ldots,$ can now be found by solving $\epsilon_2 = 0$, $\epsilon_3 = 0$ and so on. Doing this on SCRATCHPAD has resulted in the following information:

$$
\begin{aligned}
a_0 &= 1,\\
a_1 &= a_1,\\
a_2 &= a_1^2\mu(\mu+1)(3\mu+1)^2/p,\\
a_3 &= a_1^3(6\mu^4+9\mu^3-3\mu^2-6\mu-1)(3\mu+1)^4/(3pq),\\
a_4 &= a_1^4(432\mu^{11}+1872\mu^{10}+\ldots+30\mu+2)(3\mu+1)^5/3\mu p^2qr,\\
a_5 &= a_1^5(23328\mu^{14}+109836\mu^{13}+\ldots+318\mu+10)(3\mu+1)^6(\mu+1)/3p^2qrs,\\
a_6 &= a_1^6(4897760256\mu^{30}+\ldots+402\mu+6)(3\mu+1)^6/9\mu p^3q^2rst,
\end{aligned}
$$

where

$$
\begin{aligned}
p &= 12\mu^3+15\mu^2+6\mu+1,\\
q &= 24\mu^3+27\mu^2+9\mu+1,\\
r &= 24\mu^3+51\mu^2+32\mu+6,\\
s &= 216\mu^5+450\mu^4+330\mu^3+105\mu^2+15\mu+1,\\
t &= 324\mu^7+999\mu^6+918\mu^5+135\mu^4-210\mu^3-108\mu^2-18\mu-1
\end{aligned}
$$

and the large numerators are irreducible.

It can be seen that each a_i is of the form

$$a_i = a_1^i \times \text{rational function of } \mu, \tag{8}$$

where the rational function is defined over the integers. The following construction confirms that this will always be true.

Let

$$s_j = 3^{\frac{1}{2}}\sin\pi\mu j, \quad c_j = \cos\pi\mu j.$$

Then from (7)

$$s_1 = 3(2\mu+1)c_1$$

and from the normal sine and cosine formulae

$$\left.\begin{aligned} s_n &= s_{n-1}c_1 + c_{n-1}s_1\\ c_n &= c_{n-1}c_1 - \tfrac{1}{3}s_{n-1}s_1 \end{aligned}\right\} \quad \text{for} \quad n > 1.$$

For all n, then, s_n and c_n can be represented as c_1^n multiplied by some polynomial in μ. If (2) is written in terms of s_n and c_n rather than direct sines and cosines, all the $\sqrt{3}$'s vanish, and each ϵ_i in (6) will simplify to c_1^i multiplied by a polynomial, with rational coefficients, in μ and the a_j. Further examination shows that it will be linear in the highest coefficient a_j that appears, and that solving for that a_j will be consistent with $a_j \propto a_1^j$. It is thus seen that once μ has been found, by solving the transcendental equation (7), and a_1 has been fixed, finding all the other coefficients a_j is straightforward; the equations to be solved are all linear and have coefficients in the field formed by extending the rationals by μ. Repeated use of (7) has made it possible to eliminate all the trigonometric functions that at first sight appeared to be going to be involved. Subject to convergence problems, the a_j's found will, when substituted into (3), define a genuine solution to the original equation (1).

Considering the a_j as functions of μ, it is found that for $1 \leqslant j \leqslant 6$

$$a_j = a_1^j(\tfrac{3}{4}\mu)^{j-1} + \text{lower-order terms in } \mu.$$

The present analysis does not show whether this pattern continues, or consider what would be the implications if it did.

4. Cross-terms

If any one particular root of (7) is chosen for μ, the solution to (1) must be given by the sequence of a_i's found above. It is, however, possible to consider solutions to (1) corresponding to combinations of terms from several roots of (7). Consider the case of two roots σ and μ: w will have to be expressed in the form

$$w = \sum_{j,k=0}^{\infty} a_{j,k}(if)^{2(\mu j+\sigma k)}.$$

Separating into real and imaginary parts and substituting into (2) gives as the analogue to (6)

$$\sum_{j,k=0}^{\infty} \epsilon_{j,k} f^{2(\mu j+\sigma k)} = 1, \tag{9}$$

where the $\epsilon_{j,k}$ are now polynomials in σ, μ and the $a_{j,k}$. As before the constant term of this equation gives $a_{0,0} = 1$, and the terms in $f^{2\mu}$ and $f^{2\sigma}$ respectively constrain μ and σ to satisfy (7). Examination of the form of the polynomials $\epsilon_{j,k}$ shows that each involves only coefficients $a_{l,m}$ with $l \leqslant j$ and $m \leqslant k$. Furthermore $\epsilon_{j,k}$ is a linear function of $a_{j,k}$. This means that, given $a_{0,1}$ and $a_{1,0}$, all the $a_{j,k}$ can easily be found by solving the equations $\epsilon_{j,k} = 0$. As before no new transcendental quantities can be introduced by the process: each a_{jk} will be a rational function of $a_{0,1}$, $a_{1,0}$, μ and σ. For instance

$$a_{1,1} = \frac{a_{1,0}a_{0,1}(3\mu+1)(3\sigma+1)(2\mu\sigma+\mu+\sigma)}{1+3\mu\sigma+(\mu+\sigma)(2\mu\sigma+\mu+\sigma+1)}.$$

Arguments similar to those used before show that the expressions for the $a_{j,k}$ do not involve $\sqrt{3}$, and that

$$a_{j,k} = a_{1,0}^j a_{0,1}^k \times \text{rational function of } (\mu, \sigma).$$

Solutions corresponding to combinations of three or more roots of (7) behave in exactly the same way: provided that the roots μ_l of (7) considered are independent over the rationals, it is possible to compare coefficients of f to the power

$$\sum_{l=1}^{N} j_l \mu_l$$

in the expansion of (2) and obtain linear equations defining all the necessary coefficients. Again the total solution found will be determined by the values chosen for the leading coefficients $a_{0,0,\ldots,1,0,0,\ldots,0}$. It should then be possible to choose these leading coefficients to make z have desirable properties for large f. In particular it should be possible to choose them so that z, analytically continued from this expansion, satisfies

$$z \to f \quad \text{as} \quad \mathrm{Im} f \to -\infty,$$

the required deep-water condition for a steady travelling wave.

Much of the algebra that was needed in developing the results presented here was done using SCRATCHPAD. I should like to thank Dr J. H. Griesmer and Dr R. D. Jenks for making their system available to me, and assisting me in its use.

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